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# Tensegrity frameworks in one-dimensional space

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## ABSTRACT

The edge set of a graph  $G$  is partitioned into two subsets  $E_C \cup E_S$ . A tensegrity framework with underlying graph  $G$  and with cables for  $E_C$  and struts for  $E_S$  is proved to be rigidly embeddable into a one-dimensional line if and only if  $G$  is 2-edge-connected and every 2-vertex-connected component of  $G$  intersects both  $E_C$  and  $E_S$ . Polynomial algorithms are given for finding an embedding of such graphs and for checking the rigidity of a given one-dimensional embedding.

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## 1. Introduction

Tensegrity structures are pin-connected frameworks where some of the members are cables or struts. Today, tensegrity structures interest researchers in the engineering, mathematical and biological communities.

The elements of tensegrity structures, namely cables and struts, are characterized by their abilities to sustain only one type of load, while being capable of deforming freely in the opposite direction. Comparing against the regular pin-connected rod structures, the first property does not present much disadvantage, as in most cases the structures are designed so that the allowed loads induce only one type of force in each of the rods. On the other hand, the second property makes it possible to alter the geometry of the structures and thus to achieve unique technological properties. Controlling the geometry of the static structures gives rise to a variety of practical applications including producing foldable and deployable structures [13], smart structures, structures adjustable to the environmental conditions [11] and many others. Additional advantages of the tensegrity structures include significant weight reduction while not affecting the static performance and simplification of the construction process. Over the past decades, numerous studies of the advantages and the properties were performed, some of which are as follows.

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In engineering, tensegrity structures provide efficient solutions for applications like those in deployable structures [13,6], shape-controllable structures, smart sensors [12] and lightweight structures.

The biological community employs tensegrity structures as models underlying the behavior of a number of biological entities, such as the cytoskeleton [7]. Adopting such models enables biologists to interpret some observed but previously unexplained natural phenomena.

The complexity of the behavior on one hand and the special properties on the other provide the incentive for mathematical studies of tensegrity structures [2,3]. The main interest in this respect is concentrated on the issues of checking rigidity [10,5] and structural analysis of these structures.

A key problem in the design of tensegrity structures is the determination of geometrical configurations where a given structure becomes rigid. This problem, also referred to as the ‘form-finding problem’ [14], does not possess a general analytical solution, except for some special, relatively simple cases [4].

The present paper addresses a combinatorial approach for treating one-dimensional tensegrity structures, i.e. structures where all members are parallel. The paper establishes a theorem for checking the topological rigidity of these structures, i.e. deciding whether for a given graph there exists at least one rigid geometrical embedding. If there does, a graph-theoretical algorithm is provided for finding a rigid embedding for the given frame topology. This can be regarded as an alternative solution for the ‘form-finding problem’, although, for now, it is limited to one-dimensional structures. Additionally, an algorithm for checking the rigidity of a structure with a given geometry is shown to be equivalent to checking whether the corresponding graph is strongly connected.

For any rigid graph there is no one-dimensional singular embedding configuration since for the latter we need that the sum of the virtual work is equal to zero, i.e., the displacement/velocity of a joint is perpendicular to the corresponding rod, a situation that cannot arise in one-dimensional systems.

It is shown that the methodology can partly be considered as a special case of a more general theorem based on matroid theory [10], which raises the possibility that in the future the method could be expanded to multidimensional cases.

## 2. The condition for graph embeddability as a rigid one-dimensional framework

Let  $G = (V, E)$  be a finite graph with vertex set  $V$  and edge set  $E$  and let  $\chi$  denote a bipartition  $E = E_C \cup E_S$ . A function  $f : V(G) \rightarrow \mathbb{R}$  is called a *one-dimensional embedding* of  $G$  if  $x \neq y$  implies  $f(x) \neq f(y)$ .

A function  $g : V(G) \rightarrow \mathbb{R}$  satisfying

$$|g(x) - g(y)| \begin{cases} \leq |f(x) - f(y)| & \text{if } \{x, y\} \in E_C \\ \geq |f(x) - f(y)| & \text{if } \{x, y\} \in E_S, \end{cases} \quad (1)$$

and

$$\text{sign}[g(x) - g(y)] = \text{sign}[f(x) - f(y)] \quad \forall \{x, y\} \in E \quad (2)$$

is called a *motion with respect to the bipartition  $\chi$*  or for short a  $\chi$ -*motion* of the embedded graph  $G$ . Such a  $\chi$ -motion is *trivial* if there exists a constant  $c \in \mathbb{R}$  such that  $g(x) = f(x) + c$  for every  $x \in V(G)$ .

In the terminology of the real one-dimensional tensegrity structures, the vertices of a graph represent the junctions, while edges belonging to  $E_C$  and  $E_S$  correspond to cables/struts respectively. The embedding function  $f(x)$  indicates the location coordinate of junction  $x$ , while the motion function  $g(x)$  indicates the new location coordinate of junction  $x$ , after the tensegrity structure has been deformed. The requirements of Eqs. (1) and (2) are interpreted as physical constraints for the distance between end junctions of the cables and struts to become only smaller and only larger respectively, while the relative location of the two junctions remains unaltered.

A one-dimensional embedding  $f$  is called a *one-dimensional rigid embedding* of  $G$  with respect to this bipartition, or for short a *one-dimensional rigid  $\chi$ -embedding*, if every  $\chi$ -motion of it is trivial.

A circuit  $C$  of the graph  $G$  is a *mixed circuit* with respect to a bipartition  $\chi$ , or for short a  $\chi$ -*mixed circuit*, if neither  $C \cap E_C$  nor  $C \cap E_S$  is empty.

**Theorem 1.** A graph has a one-dimensional rigid  $\chi$ -embedding if and only if the graph is connected and every edge of it is contained by at least one  $\chi$ -mixed circuit.

**Remark.** Since each edge representing a rod can be replaced by a pair of edges, one representing a cable and one representing a strut, Theorem 1 essentially refers to tensegrity frameworks with all three types of elements. Observe that if a framework consists of rods only then the condition of the theorem reduces to the connectivity of the graph, a known condition described in the mathematical literature [9].

**Proof.** I. *Necessity.* The connectedness is obvious—if  $G_0$  were a connected component of a disconnected graph  $G$  then the function

$$g(x) = \begin{cases} f(x) + c_0 & \text{if } x \in V(G_0) \\ f(x) & \text{otherwise} \end{cases} \quad (3)$$

with  $c_0 \neq 0$  would be a nontrivial  $\chi$ -motion of  $G$ . Similarly, if the edge  $e = \{a, b\} \in E_S$  (or  $\in E_C$ , respectively) were a bridge of  $G$  and  $G_0$  denotes one of the components of  $G - e$  then the same function could be applied using a value of  $c_0$  such that  $|g(b) - g(a)|$  must be greater (smaller, respectively) than  $|f(b) - f(a)|$ .

Hence from now on we may suppose that  $G$  is connected and bridgeless. Consider one of its 2-connected components  $G_0$  and suppose indirectly that it has no  $\chi$ -mixed circuits, that is, all of its edges are in, say,  $E_C$ . Let  $x_0$  be a vertex of  $V(G_0)$  such that  $f(x_0)$  is an internal point of the interval spanned by the values  $\{f(v) | v \in V(G_0)\}$ . Then  $g(x) = f(x_0) + c[f(x) - f(x_0)]$  with some  $c < 1$  applied for  $x \in V(G_0)$  and then extended by an appropriate constant translation for the remaining elements of  $V(G)$  would define a nontrivial  $\chi$ -motion of  $G$ . (If all of the edges of  $G_0$  are in  $E_S$  then use the same argument with  $c > 1$ .)

II. *Sufficiency.* If every edge of a connected graph  $G$  is contained in some circuits then  $G$  is clearly bridgeless. Hence it is either 2-connected or has a cactus-decomposition into 2-connected components. It is clearly enough to prove the rigid embeddability for a single 2-connected component.

Recall that a graph is 2-vertex-connected if and only if it has no isolated vertices and for every pair of its edges there exists a circuit containing both of these edges; see, for example, Theorem 3.3.4 in [10]. Hence, if the edge set of a 2-vertex-connected graph intersects both  $E_C$  and  $E_S$  then every edge of this graph is contained in some  $\chi$ -mixed circuits.

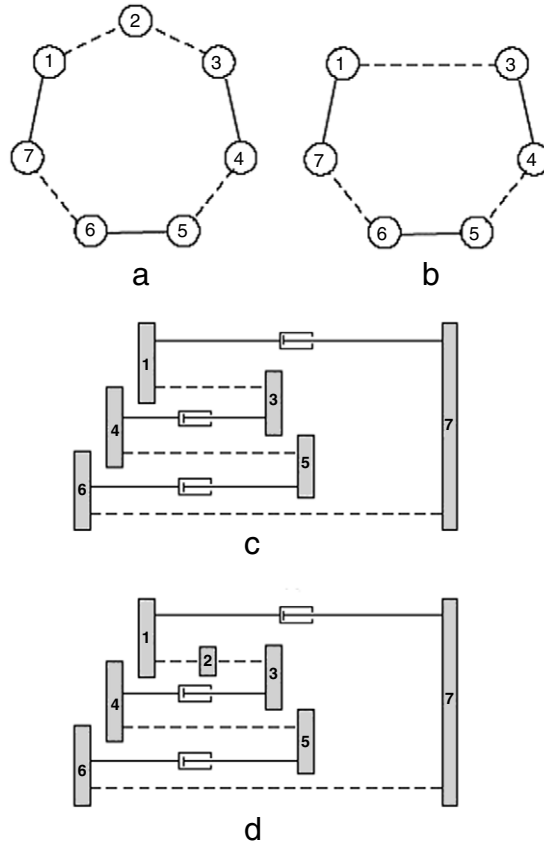
**Lemma 1.** A single  $\chi$ -mixed circuit has a one-dimensional rigid  $\chi$ -embedding.

**Proof.** We may suppose that struts and cables alternate in the circuit (otherwise temporarily replace a maximum path of struts or cables with a single strut or cable, respectively; after embedding this tensegrity framework into one-dimensional space one can readily finish the original embedding by “subdividing” some struts and cables into smaller ones). Let  $[v_0, v_1, v_2, \dots, v_{k-1}, v_k = v_0]$  be a cyclic description of the vertices of the  $\chi$ -mixed circuit. Then:

- Let  $f(v_0)$  be an arbitrary real number and  $i = 0$ .
- If  $i = k - 1$  then stop.
- If  $\{v_i, v_{i+1}\} \in E_C$  then “jump to the right”, that is, define  $f(v_{i+1})$  as an arbitrary value greater than any of the values  $f(v_0), f(v_1), \dots, f(v_i)$ .
- If  $\{v_i, v_{i+1}\} \in E_S$  then “jump to the left”, that is, define  $f(v_{i+1})$  as an arbitrary value less than any of the values  $f(v_0), f(v_1), \dots, f(v_i)$ .
- Increase the value of  $i$  by one and go to the second step.

Fig. 1 shows an example of a mixed circuit and its embedding obtained by means of this procedure.

In order to prove the rigidity of this embedding, consider a motion  $g(x)$  of the system obtained. Without loss of generality we may suppose that  $\{v_1, v_2\} \in E_S$ ; thus by Eq. (1), the following set of



**Fig. 1.** Example of a rigid embedding of a mixed circuit: (a) the mixed circuit; (b) reducing the circuit to an alternating form; (c) rigid embedding of the reduced mixed circuit; (d) the rigid embedding of the original circuit (with the corresponding cable “subdivided”).

inequalities is satisfied:

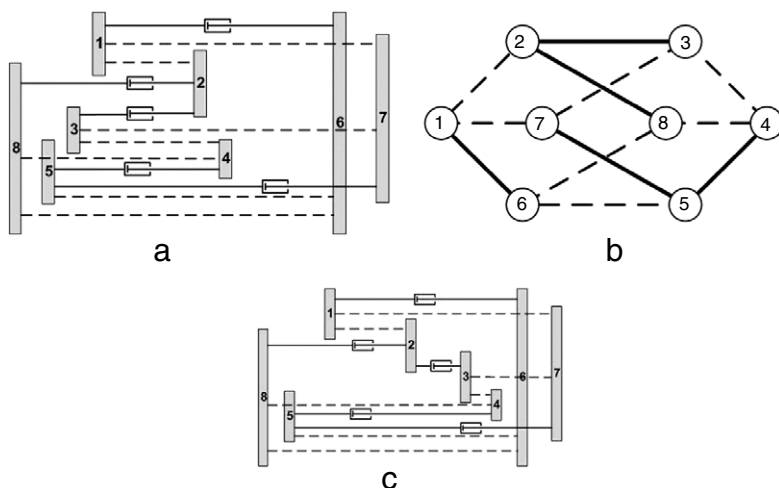
$$\begin{aligned}
 |g(v_1) - g(v_2)| &\geq |f(v_1) - f(v_2)| \\
 |g(v_2) - g(v_3)| &\leq |f(v_2) - f(v_3)| \\
 &\dots \\
 |g(v_k) - g(v_1)| &\leq |f(v_k) - f(v_1)|.
 \end{aligned} \tag{4}$$

The definition of  $g(v)$  (Eq. (2)) and the above synthesis procedure for  $\{v_i, v_j\} \in E_S$  imply that  $g(v_i) > g(v_j)$  and  $f(v_i) > f(v_j)$ , while those for  $\{v_i, v_j\} \in E_C$  imply that  $g(v_i) < g(v_j)$  and  $f(v_i) < f(v_j)$ . Therefore the above inequalities can now be rewritten without using the absolute values:

$$\begin{aligned}
 g(v_1) - g(v_2) &\geq f(v_1) - f(v_2) \\
 g(v_2) - g(v_3) &\geq f(v_2) - f(v_3) \\
 &\dots \\
 g(v_k) - g(v_1) &\geq f(v_k) - f(v_1).
 \end{aligned} \tag{5}$$

Rearranging the terms in the above inequalities yields

$$g(v_1) - f(v_1) \geq g(v_2) - f(v_2) \geq \dots \geq g(v_k) - f(v_k) \geq g(v_1) - f(v_1). \tag{6}$$



**Fig. 2.** Example of a rigid embedding of a complex graph: (a) the rigid embedding; (b) the corresponding graph; (c) example of a non-rigid embedding of the same graph.

Obviously, this set of inequalities can be resolved only if  $g(x)$  is trivial with respect to  $f(x)$ , which proves that  $f(x)$  is a rigid embedding.  $\square$

**Lemma 2.** Suppose that a 2-connected proper subgraph  $G'$  of a 2-connected graph  $G$  already has a one-dimensional rigid  $\chi$ -embedding and let  $[v_0, v_1, \dots, v_k]$  be a path of  $G$  such that  $\{v_0, v_1, \dots, v_k\} \cap V(G') = \{v_0, v_k\}$ . Then this embedding can be extended to that of a subgraph containing  $G'$  and this path. (Here  $k \geq 1$ ; hence we permit that a single edge is added only.)

**Proof.** Without loss of generality we may suppose that the edges of the path belong alternately to  $E_C$  and  $E_S$ ; see the argument in the first paragraph of the proof of Lemma 1. If  $k = 1$  then simply insert the required tensegrity element between the two end points which were already in fixed positions. If  $k > 1$  then:

- Let  $i = 0$ .
- If  $i = k - 1$  then stop.
- If  $\{v_i, v_{i+1}\} \in E_C$  then “jump to the right”, that is, define  $f(v_{i+1})$  as an arbitrary value greater than any of the values  $\{f(v_0), f(v_1), \dots, f(v_i)\} \cup \{f(v) | v \in V(G')\}$ .
- If  $\{v_i, v_{i+1}\} \in E_S$  then “jump to the left”, that is, define  $f(v_{i+1})$  as an arbitrary value less than any of the values  $\{f(v_0), f(v_1), \dots, f(v_i)\} \cup \{f(v) | v \in V(G')\}$ .
- Increase the value of  $i$  by one and go to the second step.

The rigidity of the resulting embedding can be proved in a similar fashion as it was done for Lemma 1.  $\square$

Now the proof of the sufficiency is obvious, by considering the cactus-decomposition of  $G$  and realizing the embedding of the individual 2-connected components as follows: Start with a mixed circuit as in Lemma 1 and then extend it gradually, as in Lemma 2, with new paths (including the possibility of single new edges as well). This is always possible; see, for example, the first solution of Problem 6.33 in [8].  $\square$

Fig. 2 shows an example of realizing such an embedding of a graph.

The conditions of Theorem 1 are satisfied if and only if the graph has neither bridges (which would be contained in no circuits) nor 2-vertex-connected components fully in  $E_C$  or  $E_S$ . Hence a graph satisfies these conditions if and only if it is 2-edge-connected and every 2-vertex-connected component of it intersects both  $E_C$  and  $E_S$ . Using a depth-first-search technique, one can detect both 2-vertex-connectedness and 2-edge-connectedness in linear time; see, for example, Section 5.3 of [1].

### 3. The condition for rigidity of a given one-dimensional framework

Consider a one-dimensional embedding  $F$  of a tensegrity framework. The corresponding directed graph representation  $G_F$  is defined such that the vertices  $v_i$  of  $G_F$  correspond to the joints  $i$  of  $F$ , and a tensegrity element between the joints  $i, j$  with  $f(v_i) < f(v_j)$  correspond to the edge  $e = \{i, j\}$  of  $G_F$ , with an orientation from  $i$  to  $j$  if  $e$  is a cable and from  $j$  to  $i$  if  $e$  is a strut.

By Eq. (1), a function  $g(x)$  is a valid motion function with respect to  $G_F$  if

$$g(h) - g(t) \geq f(h) - f(t) \quad \text{for every } e = (t, h) \in G_F. \quad (7)$$

**Theorem 2.** *A given one-dimensional tensegrity framework  $F$  is rigid if and only if the corresponding directed graph  $G_F$  is strongly connected.*

**Proof.** *I. Necessity.* Let us suppose indirectly that  $G_F$  possesses a directed cut-set which separates  $G_F$  into two connected subgraphs,  $G_h$  and  $G_t$ , connected respectively to the head and the tail vertices of the edges belonging to the cut-set. Then the function

$$g(x) = \begin{cases} f(x) + c_0 & \text{if } x \in G_h \\ f(x) & \text{if } x \in G_t \end{cases} \quad (8)$$

with  $c_0 \neq 0$  would be a valid nontrivial motion of  $F$ .

*II. Sufficiency.* Any two vertices  $u, v \in V(G_F)$  belong to a common directed circuit  $\{v, v_2, \dots, u, \dots, v_k, v\}$ . Applying Eq. (7) to the edges of the circuit yields a system of inequalities identical to Eq. (5). Again, this set of inequalities implies that the members and the joints corresponding to the circuit form a rigid framework not allowing relative displacement between  $u$  and  $v$ . As the condition is satisfied for any two joints of the framework, the framework as a whole is also rigid.  $\square$

Strong connectedness can also be detected in linear time; see, for example, Section 5.5 of [1].

It is interesting to note that Theorem 2 can be considered as a special case of a more general theorem developed by the first author on the basis of matroid theory. We recall Theorem 18.3.2 of [10], referring to tensegrity frameworks of any dimension.

**Theorem 3.** *Let  $F$  be a tensegrity framework and suppose that the underlying system  $F'$  is rigid (i.e. dynamically determined). Suppose that the oriented matroid  $\mathbf{M}(F)$  is graphic and is described by a directed graph  $G$ . Then  $F$  is rigid if and only if the tensegrity transformation of  $G$  is strongly connected.*

Recall that  $\mathbf{M}(F)$  in Theorem 3 is the oriented matroid represented by the row vectors of the rigidity matrix of the tensegrity framework  $F$ , and the tensegrity transformation of  $G$  reverses the orientation of the edges corresponding to struts.

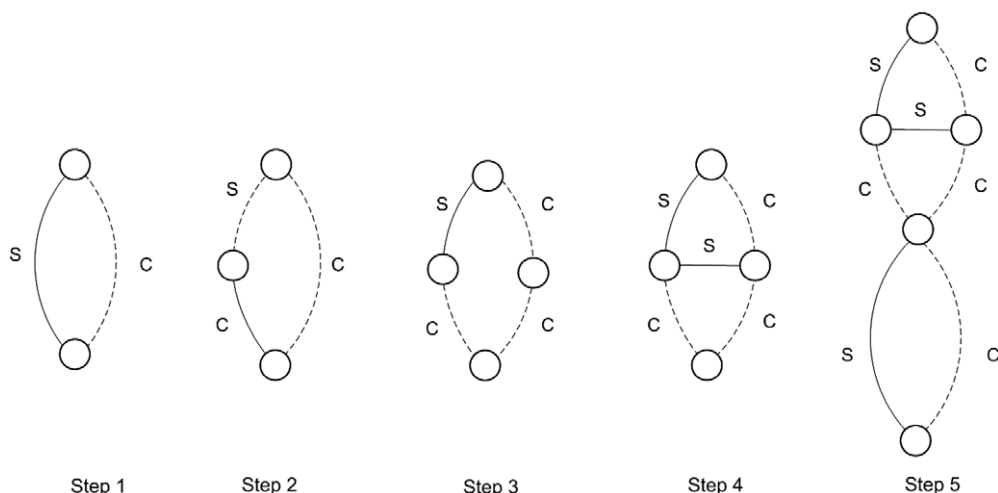
In the one-dimensional case the rigidity matrix is actually the transposed incidence matrix of  $F$ , where each column is multiplied by the length of the corresponding member. Thus, in this case,  $\mathbf{M}(F)$  is always a graphic matroid, determined by  $G_F$  itself.

As a last remark one should emphasize that, unlike in the case of bar-and-joint frameworks, if a tensegrity framework (with a fixed topology and a fixed tripartition of its edge set into  $E_R$ ,  $E_C$  and  $E_S$ ) has a rigid embedding then the set of all of its rigid embeddings is open but not necessarily dense: the complement of this set may have a positive measure. For example, joint 2 in Fig. 1(d) must be in the open interval determined by joints 1 and 3.

### 4. Deriving all one-dimensional rigid topologies

On the basis of the theorems reported in the paper it is possible to develop a method for finding all the rigid topologies by applying the construction steps, as appears below.

1. Start from the basic structure consisting of two parallel edges; one belonging to  $E_S$  and one to  $E_C$ .
2. Edge splitting: any edge can be split into two edges connected by a new vertex between them. One of the two edges is assigned to the same set as the original edge, while the second edge can be assigned arbitrarily to either  $E_S$  or  $E_C$ .



**Fig. 3.** Some topologies constructed by applying the construction steps. Each graph is obtained by applying one of the construction steps (marked below) to the previous graph.

3. Connecting vertices: Add a new edge between any two existing vertices. The new edge is arbitrarily assigned to  $E_S$  or  $E_C$ .
4. Vertex merging: having two graphs obtained through applying steps 1–4, choose an arbitrary vertex at each of the two graphs and merge them to yield a new one.

An example of applying the above construction steps appears in Fig. 3.

**Theorem 3.** A graph  $G$  possesses a one-dimensional tensegrity rigid embedding if and only if it can be obtained through applying the construction steps listed above.

*I. Sufficiency.* It can easily be verified that each construction step preserves the necessary condition for a graph being tensegrity rigid (Theorem 1), i.e., each edge is contained in at least one mixed circuit.

*II. Necessity.* First, we define a critical edge to be an edge that is the only one of its type (belonging to either  $E_S$  or to  $E_C$ ) within the 2-vertex-connected component. For a given graph  $G$ , the following reduction rules preserve the rigidity property of the graph. First we decompose the graph into 2-vertex-connected components. If there exists a vertex of degree 2 and the two edges that meet it are of the same type then replace them by one edge of that type. Otherwise, if one of the edges is critical replace the two edges by an edge of that type; otherwise there is no restriction for the replacement edge type. If all the vertices are of degree greater than 2, delete arbitrarily a non-critical edge under the condition that it preserves the 2-vertex connectivity of the component. If the reduced graph consists of only two parallel edges, one cable and one strut, then stop; the graph is rigid. Otherwise, delete these parallel edges and continue the above steps on other components until you reach the former graph. Obviously the reduction rules are the inverse to the above construction steps; thus the necessary condition of Theorem 3 is proved straightforwardly by showing that any graph satisfying Theorem 1 can be reduced to the basic structure by means of the reduction rules.

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